A Simple Formula for Scalar Curvature of Level Sets in Euclidean Spaces

Yajun Zhou

PROGRAM IN APPLIED AND COMPUTATIONAL MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544

A simple formula is derived for the Ricci scalar curvature of any smooth level set $\{\psi(x_0, x_1, \ldots, x_n) = C\}$ embedded in the Euclidean space $\mathbb{R}^{n+1}$, in terms of the gradient $\nabla\psi$ and the Laplacian $\Delta\psi$. Some applications are given to the geometry of low-dimensional p-harmonic functions and high-dimensional harmonic functions.

0 Introduction

Many problems in mathematical physics can be described by "inhomogeneous Laplace equations" in the Euclidean space $\Delta\psi(\boldsymbol{r}) = V(\boldsymbol{r})$, which not only encompass the familiar examples of linear equations (*e.g.* the Poisson equation in electrostatics, the Helmholtz equation in wave propagation, the Schrödinger equation in non-relativistic quantum mechanics) but also some important classes of nonlinear partial differential equations (including, but not limited to the Landau-Ginzburg model for superconductivity, the Gross-Pitaevski equation for Bose-Einstein condensates and the Navier-Stokes equations for viscous fluids).

In this series of works, we study the geometry of level sets associated with solutions to the "inhomogeneous Laplace equations" $\Delta\psi(\boldsymbol{r}) = V(\boldsymbol{r})$, starting with the current article which revolves around a simple formula for the Ricci scalar curvature $\mathscr{R}(\boldsymbol{r})$ at any point on a smooth level set $\{\psi(\boldsymbol{r}) = C\}$ embedded in the Euclidean space of arbitrary dimension:

$$\mathscr{R}(\boldsymbol{r}) = -\Delta\log|\nabla\psi(\boldsymbol{r})| + \nabla\cdot\left[\Delta\psi(\boldsymbol{r})\frac{\nabla\psi(\boldsymbol{r})}{|\nabla\psi(\boldsymbol{r})|^2}\right].$$

The right-hand side of the formula above invokes the gradient $\nabla\psi(\boldsymbol{r})$ and the Laplacians $\Delta\psi(\boldsymbol{r})$, $\Delta\log|\nabla\psi(\boldsymbol{r})|$ evaluated in the Cartesian coordinate system of the ambient Euclidean space. Likewise, the divergence operator $\nabla\cdot$ also refers to the flat space formulation. By convention, we will restrict our attention to cases where the gradient is non-vanishing $\nabla\psi(\boldsymbol{r}) \neq \boldsymbol{0}$, so that the appearance of $|\nabla\psi(\boldsymbol{r})|^2$ in the denominator is meaningful.

We call the proposed scalar curvature identity a "simple formula" because it only involves some familiar operations and quantities of physical interest. Especially, in many practical problems, the modulus of the gradient $|\nabla\psi(\boldsymbol{r})|$ provides a measure of the "field intensity", and $\Delta\psi(\boldsymbol{r}) = V(\boldsymbol{r})$ is given as a prescribed function in the Euclidean space or as a certain function(al) of $\psi(\boldsymbol{r})$. The simple formula in the preceding paragraph thus allows us to deduce some useful information regarding the curvature properties of the level sets and their relations to "field intensities", which we will describe in several papers in this series.

In the current work, we open with generic formulations for the geometry of level sets (§§1.1-1.2), and follow the proof of the aforementioned scalar curvature identity (§1.3) with a brief discussion on its invariance properties (§1.4). We will then show that the degenerate cases of the scalar curvature identity in low dimensions recover some familiar results in classical analysis (§2.1) and are consistent with various worked examples in classical physics (§2.2). We close this proof-of-principle article with a simple low-dimensional application to p-harmonic functions (§3) and a high-dimensional application to harmonic functions (§4).

1 A Simple Scalar Curvature Formula for Level Sets

1.1 Metric, Connection and Curvatures for Level Sets Embedded in Euclidean Spaces

A natural way to describe the local geometric properties of a smooth level set $\{\psi(x_0, x_1, \ldots, x_n) = C\}$ as a hypersurface in $\mathbb{R}^{n+1}$ is to introduce the *line element for curves on the hypersurface* as $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j$, where a Latin index takes values from 1 to n, and every repeated index implies summation [Ref. 1, §7.3, pp. 68-69]. Here, $(u^1, \ldots, u^n)$ are local curvilinear coordinates on the surface (as opposed to the Cartesian coordinates $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$), and (g_{ij}) is the (covariant) *metric tensor*. Superscript and subscript indices stand for contravariant and covariant components, respectively. Written explicitly, the components of the metric tensor takes the form $g_{ij} = \partial_i \boldsymbol{r} \cdot \partial_j \boldsymbol{r}$, where ∂_i is a short-hand for $\partial/\partial u^i$. The contravariant metric tensor (g^{ij}) is simply defined as the matrix inverse of (g_{ij}).

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The tangent vectors $\partial_i \boldsymbol{r}$ at different points on the level set in question are connected by the Gauss formula [Ref. 1, §30.4, p. 311]: $\partial_i \partial_j \boldsymbol{r} = \Gamma_{ij}^k \partial_k \boldsymbol{r} + b_{ij} \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal vector of the hypersurface, the *connection coefficients* are $\Gamma_{ij}^k = g^{k\ell} \partial_i \partial_j \boldsymbol{r} \cdot \partial_\ell \boldsymbol{r}$ and the *coefficients of second fundamental form* are $b_{ij} = \partial_i \partial_j \boldsymbol{r} \cdot \boldsymbol{n}$. The connection coefficients can also be evaluated from the Christoffel formula [Ref. 1, §29.3, p. 293]: $\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$.

The components of the Weingarten transform $\hat{W} = (b_i^j)$ is defined by $b_i^j = g^{jk} b_{ki}$ and appears in the Weingarten formula: $\partial_i \boldsymbol{n} = -b_i^j \partial_j \boldsymbol{r}$, that is, $\mathrm{d}\boldsymbol{n} = -\hat{W}\,\mathrm{d}\boldsymbol{r}$ for infinitesimal changes tangent to the surface. The *mean curvature* is equal to $\frac{1}{n}$ times the trace of the Weingarten transform: $H := \frac{1}{n} \mathrm{Tr}(\hat{W}) = \frac{1}{n} g^{ij} b_{ij}$.

The expression $R_{ij} = \partial_\ell \Gamma_{ij}^\ell - \partial_j \Gamma_{i\ell}^\ell + \Gamma_{ij}^\ell \Gamma_{\ell m}^m - \Gamma_{i\ell}^m \Gamma_{jm}^\ell$ gives the $(i,j)^{\text{th}}$ component of *Ricci tensor* on the hypersurface [Ref. 1, §37.4, p. 394], and the *Ricci scalar curvature* is defined by $\mathscr{R} = g^{ij} R_{ij}$.

For convenience, we shall borrow some notations and terminologies from physics: "force field" is defined via the negative gradient $\boldsymbol{F}(\boldsymbol{r}) := -\nabla\psi(\boldsymbol{r})$; "field intensity" $F := |\boldsymbol{F}|$ is the modulus of the gradient; "field line" or "$\boldsymbol{F}$-line" is the integral curve of the vector field $\boldsymbol{F}(\boldsymbol{r}), \boldsymbol{r} \in \mathbb{R}^{n+1}$. By convention, we orient the normal vector as $\boldsymbol{n} = \boldsymbol{F}/F$.

To extend the discussion on an isolated level set (n-dimensional manifold) to a family of hypersurfaces that fill a region in the ambient $(n+1)$-dimensional Euclidean space, we can employ the natural curvilinear coordinate system $\boldsymbol{r}(u^0, u^1, \ldots, u^n)$ as follows:

1. For every point $\boldsymbol{r}$, the zeroth component of curvilinear coordinate coincides with the "level value": $u^0 = \psi(\boldsymbol{r})$, thus the coordinate u^0 is constant on any level set;

2. Two distinct points $\boldsymbol{r}(\psi, u^1, \ldots, u^n)$ and $\boldsymbol{r}(\psi', u'^1, \ldots, u'^n)$ can be linked by the integral curve ($\boldsymbol{F}$-line) if and only if $u^j = u'^j, j = 1, \ldots, n$, so the $\boldsymbol{F}$-line acts as the curvilinear coordinate curve of u^0. Accordingly, $\boldsymbol{F} = -\nabla\psi$ is equivalent to $\partial_0 \boldsymbol{r} = -\boldsymbol{n}/F = -\boldsymbol{F}/F^2$.

From the above curvilinear coordinate system constructed in a neighborhood surrounding a non-critical point $\boldsymbol{r}$ where $F(\boldsymbol{r}) = |\nabla\psi(\boldsymbol{r})| \neq 0$, one can define *line element for curves in $(n+1)$-dimensional Euclidean space* as $\mathrm{d}\underline{s}^2 = g_{\mu\nu}\,\mathrm{d}u^\mu\,\mathrm{d}u^\nu$ where a Greek index takes values $0, 1, \ldots, n$. By definition, $g_{00} = F^{-2} = 1/g^{00}$ and $g_{0i} = g^{0i} = 0$, so the decomposition of the Euclidean metric is a direct sum of the "level value coordinate" and "level set metric". One may extend the definition of connection coefficients as $\partial_\mu \partial_\nu \boldsymbol{r} = \Gamma_{\mu\nu}^\lambda \partial_\lambda \boldsymbol{r}$, where the newly-arisen connection coefficients will be computed in the following proposition.

Proposition 1.1 (Connection Coefficients) *Suppose that a real-valued three-times continuously differentiable function $\psi \in C^3(\mathfrak{D}; \mathbb{R})$ satisfies the "inhomogeneous Laplace equation"*

$$\Delta\psi(\boldsymbol{r}) = V(\boldsymbol{r}), \quad \boldsymbol{r} \in \mathfrak{D}$$

in a certain domain $\mathfrak{D}$ of the Euclidean space $\mathbb{R}^{n+1}$, and has non-vanishing gradients therein $F(\boldsymbol{r}) := |\nabla\psi(\boldsymbol{r})| \neq 0, \boldsymbol{r} \in \mathfrak{D} \subset \mathbb{R}^{n+1}$. We endow the level sets of $\psi(\boldsymbol{r})$ with unit normal $\boldsymbol{n} = -\nabla\psi/|\nabla\psi|$ and decompose the Euclidean metric $\mathrm{d}\underline{s}^2 = g_{\mu\nu}\,\mathrm{d}u^\mu\,\mathrm{d}u^\nu = F^{-2}(\mathrm{d}\psi)^2 + g_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j$ into a direct sum of the "level value coordinate" and "level set metric", then we have the following computations for connection coefficients involving the index 0:

$$\Gamma_{ij}^0 = -F b_{ij}, \quad \Gamma_{j0}^0 = -\frac{1}{F}\frac{\partial F}{\partial u^j}, \quad \Gamma_{j0}^k = \frac{b_j^k}{F}; \tag{1}$$

$$\Gamma_{ij}^0 = -\frac{F^2}{2}\frac{\partial g_{ij}}{\partial\psi}, \quad \Gamma_{00}^j = -\frac{1}{2}g^{jm}\frac{\partial g_{00}}{\partial u^m} = \frac{1}{F^3}g^{jm}\frac{\partial F}{\partial u^m}; \tag{2}$$

$$\Gamma_{00}^0 = \frac{1}{2}g^{00}\partial_0 g_{00} = -\frac{\partial}{\partial\psi}\log F, \tag{3}$$

along with a modified version of the "harmonic coordinate condition":

$$\Gamma^0 := g^{\mu\nu}\Gamma_{\mu\nu}^0 = F^2(\Gamma_{00}^0 - \Gamma_{m0}^m) = -V. \tag{4}$$

Proof To prove the three identities in Eq. 1, it would suffice to compare the equation $\partial_\mu \partial_\nu \boldsymbol{r} = \Gamma_{\mu\nu}^\lambda \partial_\lambda \boldsymbol{r}$ with the Gauss and Weingarten formulae:

$$\frac{\partial^2 \boldsymbol{r}}{\partial u^i \partial u^j} = \Gamma_{ij}^k \frac{\partial \boldsymbol{r}}{\partial u^k} - F b_{ij}\frac{\partial \boldsymbol{r}}{\partial\varphi}, \qquad \frac{\partial}{\partial u^j}\left(F\frac{\partial \boldsymbol{r}}{\partial\varphi}\right) = b_j^k \frac{\partial \boldsymbol{r}}{\partial u^k}.$$

The two identities in Eq. 2, as well as Eq. 3, follow from the Christoffel formula $\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\eta}(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\mu\eta} - \partial_\eta g_{\mu\nu})$.

Juxtaposing the two expressions of Γ^0_{ij} in Eqs. 1 and 2, we can put down

$$\frac{\partial g_{ij}}{\partial \psi} = \frac{2}{F} b_{ij}, \quad \frac{\partial \log \det(g_{ij})}{\partial \psi} = g^{ij} \frac{\partial g_{ij}}{\partial \psi} = \frac{2}{F} g^{ij} b_{ij} = \frac{2nH}{F} = \frac{2\operatorname{Tr}(\hat{W})}{F}. \tag{5}$$

On the other hand, the "inhomogeneous Laplace equation" $\Delta \psi(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{F}(\boldsymbol{r}) = V(\boldsymbol{r})$ prescribes the divergence of force field $\boldsymbol{F}(\boldsymbol{r})$, i.e. $(F/\sqrt{g})\partial_0(F\sqrt{g}) = V$ (hereafter $g = \det(g_{ij})$), which can be combined with Eqs. 1 and 5 into

$$\Gamma^0 := g^{00}\Gamma^0_{00} + g^{ij}\Gamma^0_{ij} = F^2(\Gamma^0_{00} - \Gamma^m_{m0}) = -\frac{F}{\sqrt{g}}\frac{\partial}{\partial \psi}(F\sqrt{g}) = -\nabla \cdot \nabla \psi = -V,$$

as stated in Eq. 4. If $V(\boldsymbol{r}) \equiv 0$, and $\psi(\boldsymbol{r})$ is a harmonic function, then the formula $\Gamma^0 = g^{\mu\nu}\Gamma^0_{\mu\nu} = 0$ hearkens back to the "harmonic coordinate condition" in general relativity. ∎

Remark From the identity $\partial_0 g_{ij} = 2b_{ij}/F$, we can also readily deduce $\partial_0 g^{ij} = -2g^{ik}b^j_k/F$.

Another by-product of the foregoing argument is the following result that will be used later in this work:

$$\operatorname{Tr}(\hat{W}) + \frac{\partial F}{\partial \psi} = \frac{V}{F}, \quad \text{i.e. } (\boldsymbol{n} \cdot \nabla)\log F = \operatorname{Tr}(\hat{W}) - \frac{V}{F}. \tag{6}$$

When $V = 0$ and $n = 2$, the formula above is a standard exercise in electrostatics [Ref. 2, Exercise 1.11]. □

1.2 Laplace Operator in Curvilinear Coordinates and Evolution of Mean Curvature

The Laplace operator on the Euclidean space $\mathbb{R}^{n+1}$ can be presented in curvilinear coordinates as

$$\Delta = g^{\mu\nu}(\partial_\mu \partial_\nu - \Gamma^\lambda_{\mu\nu}\partial_\lambda) = \frac{1}{\sqrt{\det(g_{\mu\nu})}}\partial_\lambda\left(g^{\lambda\eta}\sqrt{\det(g_{\mu\nu})}\partial_\eta\right).$$

Here, $\det(g_{\mu\nu}) = g/F^2$. Similarly, one can define the *Laplace operator on the level set* Σ as

$$\Delta_\Sigma = g^{ij}(\partial_i \partial_j - \Gamma^k_{ij}\partial_k) = \frac{1}{\sqrt{g}}\partial_k\left(g^{k\ell}\sqrt{g}\partial_\ell\right).$$

Proposition 1.2 (Decomposition of Laplacian) *The Laplace operator Δ can be rewritten as*

$$\Delta = \Delta_\Sigma + F^2 \frac{\partial^2}{\partial \psi^2} - \frac{1}{F} g^{jm} \frac{\partial F}{\partial u^m} \frac{\partial}{\partial u^j} + V \frac{\partial}{\partial \psi}, \tag{7}$$

where

$$\Delta_\Sigma := \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j) = g^{ij}(\partial_i \partial_j - \Gamma^k_{ij}\partial_k)$$

is the Laplace-Beltrami operator on the level set of ψ. Accordingly, we have the following formula

$$\Delta \log F = -F\Delta_\Sigma \frac{1}{F} + F^2 \frac{\partial^2 \log F}{\partial \psi^2} + \Gamma^0 \Gamma^0_{00} = -F\Delta_\Sigma \frac{1}{F} + F^2 \frac{\partial^2 \log F}{\partial \psi^2} - \frac{\operatorname{Tr}(\hat{W})V}{F} + \frac{V^2}{F^2}. \tag{8}$$

Proof By definition, we have

$$\Delta = \Delta_\Sigma - g^{ij}\Gamma^0_{ij}\frac{\partial}{\partial \psi} + g^{00}\left(\frac{\partial^2}{\partial \psi^2} - \Gamma^j_{00}\frac{\partial}{\partial u^j} - \Gamma^0_{00}\frac{\partial}{\partial \psi}\right) = \Delta_\Sigma + g^{00}\left(\frac{\partial^2}{\partial \psi^2} - \Gamma^j_{00}\frac{\partial}{\partial u^j}\right) - \Gamma^0 \frac{\partial}{\partial \psi}.$$

With the substitution of $g^{00} = F^2$ and the expressions for $\Gamma^0_{ij}, \Gamma^j_{00}, \Gamma^0_{00}, \Gamma^0$ from Proposition 1.1, we obtain the claimed result in Eq. 7. To derive Eq. 8, it would suffice to spell out the term $\Gamma^0 \Gamma^0_{00}$ using Eq. 6. ∎

Remark By the relations $\Delta \boldsymbol{r} = \boldsymbol{0}$ and $\Delta_\Sigma \boldsymbol{r} = \operatorname{Tr}(\hat{W})\boldsymbol{n}$, one may also use Eq. 7 to deduce an explicit expression for the second order derivative $\partial^2 \boldsymbol{r}/\partial \psi^2$. □

For a hypersurface embedded in Euclidean space, its Ricci tensor (R_{ij}) and Ricci scalar curvature $\mathscr{R}$ can be spelt out "extrinsically" in terms of the Weingarten transformation and the principal curvatures. Concretely speaking, in the principal curvature coordinate system where the Weingarten transformation $\hat{W} = (b_j^i)$ is represented by a diagonal matrix with eigenvalues $k_1, \ldots, k_n$ (*viz.* the n principal curvatures of the hypersurface), one has

$$(R_{ij})_{1 \leq i,j \leq n} = \begin{pmatrix} g_{11}[(k_1 + \cdots + k_n)k_1 - k_1^2] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & g_{nn}[(k_1 + \cdots + k_n)k_n - k_n^2] \end{pmatrix} = (b_k^k b_{ij} - b_{ki} g^{km} b_{mj})_{1 \leq i,j \leq n}.$$

Here, to identify the two extreme ends of the equation above, we may recall that the geometric relation

$$b_{ki} g^{km} b_{mj} - b_k^k b_{ij} = -R_{ij}$$

applies to any curvilinear coordinate system on any n-dimensional hypersurface embedded in $\mathbb{R}^{n+1}$, as evident from the contraction $R_{ij} = R_{ikj}^k$ of the Riemann curvature tensor $R_{i\ell j}^k = b_{ij} b_\ell^k - b_{i\ell} b_j^k$. Consequently, the Ricci scalar curvature $\mathscr{R} = g^{ij} R_{ij} = b_i^i b_j^j - b_j^i b_i^j$ can be explicitly given in terms of moments of principal curvatures $\mathscr{R} = (\sum_{\ell=1}^n k_\ell)^2 - \sum_{m=1}^n k_m^2 = [\mathrm{Tr}(\hat{W})]^2 - \mathrm{Tr}(\hat{W}^2)$.

In the proposition below, both the surface Laplacian Δ_Σ and the Ricci scalar curvature $\mathscr{R}$ turn up in the evolution of the mean curvature along the "level value coordinate".

Proposition 1.3 (Evolution of the Second Fundamental Form) *We have the following identities*

$$\frac{\partial b_{ij}}{\partial \psi} = (b_j^k b_{ki} - \partial_i \partial_j + \Gamma_{ij}^k \partial_k) \frac{1}{F} \tag{9}$$

and

$$\frac{\partial \mathrm{Tr}(\hat{W})}{\partial \psi} = -\Delta_\Sigma \frac{1}{F} - \frac{\mathrm{Tr}(\hat{W}^2)}{F} = -\Delta_\Sigma \frac{1}{F} - \frac{[\mathrm{Tr}(\hat{W})]^2 - \mathscr{R}}{F}, \tag{10}$$

where $\mathscr{R}$ is the Ricci scalar curvature of the level set in question.

Proof From the identity $\partial_0(\partial_i \partial_j \boldsymbol{r}) = \partial_i(\partial_0 \partial_j \boldsymbol{r})$, we may deduce $\partial_0 \Gamma_{ij}^0 + \Gamma_{ij}^\nu \Gamma_{\nu 0}^0 = \partial_i \Gamma_{j0}^0 + \Gamma_{j0}^\nu \Gamma_{\nu i}^0$. This results in Eq. 9, upon substitution of the connection coefficients. Combining $\mathrm{Tr}(\hat{W}) = g^{ij} b_{ij}$ and $\partial_0 g^{ij} = -2g^{ik} b_k^j / F$ with Eq. 9, we obtain

$$\frac{\partial \mathrm{Tr}(\hat{W})}{\partial \psi} = -\Delta_\Sigma \frac{1}{F} - \frac{b_j^k b_k^j}{F}, \quad \text{where } b_j^k b_k^j = \mathrm{Tr}(\hat{W}^2) = [\mathrm{Tr}(\hat{W})]^2 - \mathscr{R}.$$

This verifies Eq. 10. ∎

Remark We originally discovered the scalar curvature identity mentioned in the introduction by an examination of the evolution equation for mean curvature (Eq. 10). We leave it to the interested readers to recover such an "extrinsic proof". In the next subsection, we will only describe an "intrinsic approach" based on the definition of Ricci scalar curvature via the metric and connection on the level set. As will be explained elsewhere, the intrinsic method has better extendibility when the ambient space is not Euclidean, and/or the co-dimension of the submanifold is higher than one. □

1.3 Scalar Curvature Identity and Evolution of Ricci Curvature

We are now ready to prove the formula

$$\mathscr{R}(\boldsymbol{r}) = -\Delta \log |\nabla \psi(\boldsymbol{r})| + \nabla \cdot \left[\Delta \psi(\boldsymbol{r}) \frac{\nabla \psi(\boldsymbol{r})}{|\nabla \psi(\boldsymbol{r})|^2} \right]$$

mentioned in the introduction.

Proposition 1.4 (Scalar Curvature Identity) *Let $F = |\nabla \psi|$ and $V = \Delta \psi$, then we have the following geometric identity for the scalar curvature of the level sets for ψ:*

$$\Delta \log F + \mathscr{R} - \frac{\partial V}{\partial \psi} - \frac{2 \mathrm{Tr}(\hat{W}) V}{F} + \frac{V^2}{F^2} = \Delta \log F + \mathscr{R} - F^2 \frac{\partial}{\partial \psi} \frac{V}{F^2} - \frac{V^2}{F^2} = \Delta \log F + \mathscr{R} - \nabla \cdot \left(\frac{V \nabla \psi}{F^2} \right) = 0. \tag{11}$$

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Proof We define the Ricci tensor for the local curvilinear coordinates $(u^0 = \psi, u^1, \ldots, u^n)$ by

$$\underline{R}_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\tau_{\lambda\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\lambda_{\nu\tau},$$

so that $g^{\mu\nu}\underline{R}_{\mu\nu} = 0$ follows from the flatness of the Euclidean space $\mathbb{R}^{n+1}$. Meanwhile, by the direct sum decomposition of the metric $g_{\mu\nu}$, the following identity holds

$$0 = g^{\mu\nu}\underline{R}_{\mu\nu} = F^2(\partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{00}\Gamma^\tau_{\lambda\tau} - \Gamma^\tau_{0\lambda}\Gamma^\lambda_{0\tau}) + \mathscr{R} + g^{ij}(\partial_0 \Gamma^0_{ij} - \partial_j \Gamma^0_{i0} + \Gamma^\lambda_{ij}\Gamma^0_{\lambda 0} + \Gamma^0_{ij}\Gamma^m_{0m} - \Gamma^0_{i\lambda}\Gamma^\lambda_{j0} - \Gamma^m_{i0}\Gamma^0_{jm}). \quad (12)$$

Using the modified version of the harmonic coordinate condition $\Gamma^0 := g^{\mu\nu}\Gamma^0_{\mu\nu} = -V$ (Eq. 4), we may derive

$$F^2\Gamma^\lambda_{00}\Gamma^\tau_{\lambda\tau} + g^{ij}(\Gamma^\lambda_{ij}\Gamma^0_{\lambda 0} + \Gamma^0_{ij}\Gamma^m_{0m}) = F^2\Gamma^m_{00}\Gamma^\tau_{m\tau} + g^{ij}\Gamma^m_{ij}\Gamma^0_{m0} + \Gamma^0(\Gamma^0_{00} + \Gamma^m_{0m}) = F^2\Gamma^m_{00}\Gamma^\tau_{m\tau} + g^{ij}\Gamma^m_{ij}\Gamma^0_{m0} - V(\Gamma^0_{00} + \Gamma^m_{0m}).$$

Combining this result with the identity $\partial_0 \Gamma^0_{ij} + \Gamma^\nu_{ij}\Gamma^0_{\nu 0} = \partial_i \Gamma^0_{j0} + \Gamma^\nu_{j0}\Gamma^0_{\nu i}$, we may rewrite Eq. 12 as

$$0 = F^2(\partial_m \Gamma^m_{00} - \partial_0 \Gamma^m_{0m} + \Gamma^m_{00}\Gamma^\tau_{m\tau} - \Gamma^\tau_{0\lambda}\Gamma^\lambda_{0\tau}) + \mathscr{R} + g^{ij}(\Gamma^\nu_{i0}\Gamma^0_{jv} - \Gamma^\nu_{ij}\Gamma^0_{v0} + \Gamma^m_{ij}\Gamma^0_{m0} - \Gamma^0_{i\lambda}\Gamma^\lambda_{j0} - \Gamma^m_{i0}\Gamma^0_{jm}) - V(\Gamma^0_{00} + \Gamma^m_{0m})$$

$$= F^2(\partial_m \Gamma^m_{00} - \partial_0 \Gamma^m_{0m} + \Gamma^m_{00}\Gamma^\tau_{m\tau} - \Gamma^\tau_{0\lambda}\Gamma^\lambda_{0\tau}) + \mathscr{R} - g^{ij}(\Gamma^0_{ij}\Gamma^0_{00} + \Gamma^0_{im}\Gamma^m_{j0}) - V(\Gamma^0_{00} + \Gamma^m_{0m})$$

$$= F^2(\partial_m \Gamma^m_{00} - \partial_0 \Gamma^m_{0m} + \Gamma^m_{00}\Gamma^\tau_{m\tau} - \Gamma^m_{0\lambda}\Gamma^\lambda_{0m} - \Gamma^0_{0m}\Gamma^m_{00}) + \mathscr{R} - g^{ij}\Gamma^0_{im}\Gamma^m_{j0} - V\Gamma^m_{0m}$$

$$= F^2\left[\frac{\partial^2}{\partial\psi^2}\log F - \partial_0(\Gamma^m_{0m} - \Gamma^0_{00})\right] + F^2(\partial_m \Gamma^m_{00} + \Gamma^m_{00}\Gamma^\ell_{m\ell} - \Gamma^0_{0m}\Gamma^m_{00}) + \mathscr{R} - V\Gamma^m_{0m}$$

$$= F^2\left[\frac{\partial^2}{\partial\psi^2}\log F - \frac{\partial}{\partial\psi}\frac{V}{F^2}\right] + F^2(\partial_m \Gamma^m_{00} + \Gamma^m_{00}\Gamma^\ell_{m\ell} - \Gamma^0_{0m}\Gamma^m_{00}) + \mathscr{R} - V\Gamma^m_{0m},$$

where we have employed the relation $F^2\Gamma^m_{0\ell}\Gamma^\ell_{0m} + g^{ij}\Gamma^0_{im}\Gamma^m_{j0} = 0$ in the penultimate step. Exploiting the identities itemized in Proposition 1.1, along with the relation $\Gamma^\ell_{m\ell} = \partial_m \log\sqrt{g}$, we may reduce Eq. 12 into

$$0 = F^2\frac{\partial^2}{\partial\psi^2}\log F - \frac{\partial V}{\partial\psi} + F^2\left[\frac{\partial}{\partial u^m}\left(-\frac{1}{F}g^{jm}\frac{\partial}{\partial u^j}\frac{1}{F}\right) + \frac{1}{F^3}g^{jm}\Gamma^\ell_{m\ell}\frac{\partial F}{\partial u^j} + \frac{1}{F^2}g^{\ell m}\frac{\partial\log F}{\partial u^\ell}\frac{\partial\log F}{\partial u^m}\right] - V(\Gamma^m_{0m} + 2\Gamma^0_{00}) + \mathscr{R}$$

$$= F^2\frac{\partial^2}{\partial\psi^2}\log F - \frac{\partial V}{\partial\psi} + F^2\left[-\frac{1}{F}\frac{\partial}{\partial u^m}\left(g^{jm}\frac{\partial}{\partial u^j}\frac{1}{F}\right) - \frac{1}{F}\frac{g^{jm}}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^m}\frac{\partial}{\partial u^j}\frac{1}{F}\right] - V(\Gamma^m_{0m} + 2\Gamma^0_{00}) + \mathscr{R}$$

$$= -F\Delta_\Sigma \frac{1}{F} + F^2\frac{\partial^2}{\partial\psi^2}\log F - \frac{\partial V}{\partial\psi} - V(\Gamma^m_{0m} + 2\Gamma^0_{00}) + \mathscr{R} \stackrel{\text{Eq. 8}}{=\!=\!=} \Delta\log F - \frac{\partial V}{\partial\psi} - V(\Gamma^m_{0m} + \Gamma^0_{00}) + \mathscr{R},$$

which completes the proof. ∎

Remark In Eq. 11, the expression $\partial V/\partial\psi$ is understood as the derivative with respect to the curvilinear coordinate ψ, thus is well-defined for any continuously differentiable inputs $V \in C^1(\mathfrak{D};\mathbb{R})$. □

Proposition 1.5 (Evolution of Ricci Curvature) *We have the following formulae valid in arbitrary dimensions:*

$$\frac{\partial R_{ij}}{\partial\psi} = -b_{ij}\Delta_\Sigma\frac{1}{F} + \frac{\mathscr{R}b_{ij} - b^\ell_\ell R_{ij}}{F} + b^\ell_\ell\left(-\partial_i\partial_j\frac{1}{F} + \Gamma^k_{ij}\partial_k\frac{1}{F}\right) + b^k_j\left(\partial_i\partial_k\frac{1}{F} - \Gamma^\ell_{ik}\partial_\ell\frac{1}{F}\right) + b^k_i\left(\partial_j\partial_k\frac{1}{F} - \Gamma^\ell_{jk}\partial_\ell\frac{1}{F}\right); \quad (13)$$

$$\frac{\partial\mathscr{R}}{\partial\psi} = -\frac{2b^j_k R^k_j}{F} - 2b^\ell_\ell\Delta_\Sigma\frac{1}{F} + 2g^{ij}b^k_j\left(\partial_i\partial_k\frac{1}{F} - \Gamma^\ell_{ik}\partial_\ell\frac{1}{F}\right) \quad (14)$$

$$= -\frac{2G^{ij}b_{ij}}{F} - \frac{b^\ell_\ell\mathscr{R}}{F} - \frac{2}{\sqrt{g}}\partial_i\left(\beta^{ik}\sqrt{g}\partial_k\frac{1}{F}\right), \quad (14^*)$$

where $R^k_j := g^{ik}R_{ij}, \beta^{ik} := b^\ell_\ell g^{ik} - g^{ij}b^k_j$ and $G^{ij} = R^{ij} - \frac{1}{2}g^{ij}\mathscr{R}$ is the (contravariant) Einstein tensor with $R^{ij} = g^{ik}R^j_k$.

Proof Using Eqs. 9 and 10 from Proposition 1.3, we may differentiate both sides of the identity $R_{ij} = b^k_k b_{ij} - b_{ki}g^{km}b_{mj}$

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to obtain

$$\frac{\partial R_{ij}}{\partial \psi} = \left(-\Delta_\Sigma \frac{1}{F} - \frac{b_j^i b_i^j}{F}\right) b_{ij} + b_\ell^\ell \left(\frac{b_{ki} g^{km} b_{mj}}{F} - \partial_i \partial_j \frac{1}{F} + \Gamma_{ij}^k \partial_k \frac{1}{F}\right)$$

$$+ \frac{2 b_{ki} g^{k\ell} b_\ell^m b_{mj}}{F} - \left(\frac{b_{\ell i} g^{\ell p} b_{pk}}{F} - \partial_i \partial_k \frac{1}{F} + \Gamma_{ik}^\ell \partial_\ell \frac{1}{F}\right) g^{km} b_{mj} - \left(\frac{b_{\ell j} g^{\ell p} b_{pk}}{F} - \partial_j \partial_k \frac{1}{F} + \Gamma_{jk}^\ell \partial_\ell \frac{1}{F}\right) g^{km} b_{mi}$$

$$= -b_{ij} \Delta_\Sigma \frac{1}{F} + \frac{\mathscr{R} - b_\ell^\ell b_k^k}{F} b_{ij} + b_\ell^\ell \left(\frac{b_k^k b_{ij} - R_{ij}}{F} - \partial_i \partial_j \frac{1}{F} + \Gamma_{ij}^k \partial_k \frac{1}{F}\right)$$

$$+ \frac{2 b_{ki} g^{k\ell} (b_m^m b_{\ell j} - R_{\ell j})}{F} - \left(\frac{b_m^m b_{ik} - R_{ik}}{F} - \partial_i \partial_k \frac{1}{F} + \Gamma_{ik}^\ell \partial_\ell \frac{1}{F}\right) b_j^k - \left(\frac{b_m^m b_{jk} - R_{jk}}{F} - \partial_j \partial_k \frac{1}{F} + \Gamma_{jk}^\ell \partial_\ell \frac{1}{F}\right) b_i^k$$

$$= -b_{ij} \Delta_\Sigma \frac{1}{F} + \frac{\mathscr{R} b_{ij} - b_\ell^\ell R_{ij}}{F} + b_\ell^\ell \left(-\partial_i \partial_j \frac{1}{F} + \Gamma_{ij}^k \partial_k \frac{1}{F}\right) + b_j^k \left(\partial_i \partial_k \frac{1}{F} - \Gamma_{ik}^\ell \partial_\ell \frac{1}{F}\right) + b_i^k \left(\partial_j \partial_k \frac{1}{F} - \Gamma_{jk}^\ell \partial_\ell \frac{1}{F}\right),$$

thus proving Eq. 13. (Here, we have used the relation $b_i^m b_{mj} = b_{ki} g^{km} b_{mj} = b_j^k b_{ki}$, which also implies the symmetry $b_i^\ell R_{\ell j} = b_j^\ell R_{\ell i}$.)

Carrying this further, we have

$$\frac{\partial \mathscr{R}}{\partial \psi} = -\frac{2 g^{ik} b_k^j R_{ij}}{F} + g^{ij} \frac{\partial R_{ij}}{\partial \psi} = -\frac{2 b_k^j R_j^k}{F} - 2 b_\ell^\ell \Delta_\Sigma \frac{1}{F} + 2 g^{ij} b_j^k \left(\partial_i \partial_k \frac{1}{F} - \Gamma_{ik}^\ell \partial_\ell \frac{1}{F}\right),$$

as claimed in Eq. 14. Using covariant derivatives indicated by semicolons, we may rewrite the right-hand side of Eq. 14 as

$$-\frac{2 b_k^j R_j^k}{F} - 2 \beta^{ik} \left(\frac{1}{F}\right)_{;ik} = -\frac{2 b_k^j R_j^k}{F} - 2 \left(\beta^{ik} \partial_i \frac{1}{F}\right)_{;k} + 2 (\beta^{ik})_{;k} \partial_i \frac{1}{F} = -\frac{2 b_k^j R_j^k}{F} - \frac{2}{\sqrt{g}} \partial_i \left(\beta^{ik} \sqrt{g} \partial_k \frac{1}{F}\right) + 2 (\beta^{ik})_{;k} \partial_i \frac{1}{F},$$

but

$$(\beta^{ik})_{;k} = b_{\ell;k}^\ell g^{ik} - g^{ij} b_{j;k}^k = b_{\ell;k}^\ell g^{ik} - g^{ij} b_{k;j}^k = 0. \tag{15}$$

This leads to a succinct proof of Eq. 14*, based on the fact that covariant derivatives of the metric vanishes $(g^{ik})_{;\ell} := \partial_\ell g^{ik} + g^{im} \Gamma_{m\ell}^k + g^{km} \Gamma_{m\ell}^i = 0$, and the symmetry $(\partial_i b_j^k + \Gamma_{i\ell}^k b_j^\ell - \Gamma_{ij}^\ell b_\ell^k =:) b_{j;i}^k = b_{i;j}^k$ which follows from the Codazzi-Mainardi equation $\partial_i b_j^k + \Gamma_{i\ell}^k b_j^\ell = \partial_j b_i^k + \Gamma_{j\ell}^k b_i^\ell$. ∎

1.4 Diffeomorphic Invariance of the Curvature Formulae

Locally, a level set of ψ can be identified with a level set of $f \circ \psi$, where the smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has non-vanishing derivative $f' \neq 0$. It is thus natural to ask if the formulae for mean and scalar curvatures (Eqs. 6 and 11) would remain intact under such function compositions. This question is answered in the positive by the following proposition.

Proposition 1.6 (Diffeomorphic Invariance) *For every $f \in C^3(\mathbb{R}; \mathbb{R})$ satisfying $f' \neq 0$, Eq. 6 is diffeomorphically invariant in the sense of*

$$\frac{df/d\psi}{|df/d\psi|} \mathrm{Tr}(\hat{W}) + \frac{\partial |\nabla(f \circ \psi)|}{\partial (f \circ \psi)} = \frac{\Delta(f \circ \psi)}{|\nabla(f \circ \psi)|} \tag{16}$$

where $\mathrm{Tr}(\hat{W})$ is evaluated with respect to the normal $\boldsymbol{n} = -\nabla\psi/|\nabla\psi|$, and the diffeomorphic invariance of Eq. 11 is manifested in the following way:

$$\Delta \log |\nabla(f \circ \psi)| + \mathscr{R} - \nabla \cdot \left[\Delta(f \circ \psi) \frac{\nabla(f \circ \psi)}{|\nabla(f \circ \psi)|^2}\right] = 0. \tag{17}$$

Proof By the chain rule, we have

$$\nabla(f \circ \psi) = \frac{df}{d\psi} \nabla\psi, \quad \Delta(f \circ \psi) = \frac{df}{d\psi} \Delta\psi + \frac{d^2 f}{d\psi^2} |\nabla\psi|^2,$$

thus we may compute

$$\frac{\Delta(f\circ\psi)}{|\nabla(f\circ\psi)|} - \frac{\partial|\nabla(f\circ\psi)|}{\partial(f\circ\psi)} = \frac{\mathrm{d}f/\mathrm{d}\psi}{|\mathrm{d}f/\mathrm{d}\psi|}\frac{\Delta\psi}{|\nabla\psi|} + \frac{\mathrm{d}^2f/\mathrm{d}\psi^2}{|\mathrm{d}f/\mathrm{d}\psi|}|\nabla\psi| - \frac{1}{\mathrm{d}f/\mathrm{d}\psi}\frac{\partial}{\partial\psi}\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\nabla\psi\right| = \frac{\mathrm{d}f/\mathrm{d}\psi}{|\mathrm{d}f/\mathrm{d}\psi|}\left(\frac{\Delta\psi}{|\nabla\psi|} - \frac{\partial|\nabla\psi|}{\partial\psi}\right),$$

which confirms Eq. 16 through Eq. 6. Next, we evaluate

$$\nabla\cdot\left[\Delta(f\circ\psi)\frac{\nabla(f\circ\psi)}{|\nabla(f\circ\psi)|^2}\right] - \Delta\log|\nabla(f\circ\psi)| = \nabla\cdot\left[\left(\frac{\mathrm{d}f}{\mathrm{d}\psi}\Delta\psi + \frac{\mathrm{d}^2f}{\mathrm{d}\psi^2}|\nabla\psi|^2\right)\frac{\frac{\mathrm{d}f}{\mathrm{d}\psi}\nabla\psi}{\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\nabla\psi\right|^2}\right] - \Delta\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\nabla\psi\right|$$

$$= \nabla\cdot\left[\left(\Delta\psi + \frac{\mathrm{d}^2f/\mathrm{d}\psi^2}{\mathrm{d}f/\mathrm{d}\psi}|\nabla\psi|^2\right)\frac{\nabla\psi}{|\nabla\psi|^2}\right] - \Delta\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\nabla\psi\right|$$

$$= \nabla\cdot\left(\Delta\psi\frac{\nabla\psi}{|\nabla\psi|^2}\right) - \Delta\log|\nabla\psi| + \nabla\cdot\left(\frac{\mathrm{d}^2f/\mathrm{d}\psi^2}{\mathrm{d}f/\mathrm{d}\psi}\nabla\psi\right) - \Delta\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right|,$$

and quote the decomposition of Laplacian (Eq. 7) to deduce

$$\nabla\cdot\left(\frac{\mathrm{d}^2f/\mathrm{d}\psi^2}{\mathrm{d}f/\mathrm{d}\psi}\nabla\psi\right) - \Delta\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| = |\nabla\psi|^2\frac{\mathrm{d}^2}{\mathrm{d}\psi^2}\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| + (\Delta\psi)\frac{\mathrm{d}}{\mathrm{d}\psi}\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| - \Delta\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right|$$

$$= |\nabla\psi|^2\frac{\mathrm{d}^2}{\mathrm{d}\psi^2}\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| + (\Delta\psi)\frac{\mathrm{d}}{\mathrm{d}\psi}\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| - \left[|\nabla\psi|^2\frac{\partial^2}{\partial\psi^2} + (\Delta\psi)\frac{\partial}{\partial\psi}\right]\log\left|\frac{\mathrm{d}f}{\mathrm{d}\psi}\right| = 0,$$

which completes the verification of Eq. 17. ∎

2 Scalar Curvature Formulae in Low Dimensions

2.1 Examples from Classical Analysis

The Ricci scalar curvature identity (Eq. 11) in low dimensions can also be independently verified with brute force.

Example 2.1 (Ricci Scalar Curvature in Low Dimensions) *For any real-valued, three-times continuously differentiable function $\psi\in C^3(\Omega;\mathbb{R})$ defined in a two-dimensional domain $\Omega\subset\mathbb{R}^2$, one has the exact identity*

$$(\partial_x^2+\partial_y^2)\log\sqrt{(\partial_x\psi)^2+(\partial_y\psi)^2} = \partial_x\left[\frac{(\partial_x^2\psi+\partial_y^2\psi)\partial_x\psi}{(\partial_x\psi)^2+(\partial_y\psi)^2}\right] + \partial_y\left[\frac{(\partial_x^2\psi+\partial_y^2\psi)\partial_y\psi}{(\partial_x\psi)^2+(\partial_y\psi)^2}\right], \quad wherever\ |\nabla\psi|\neq 0, \tag{18}$$

which degenerates into the obvious relation $\partial_x^2\log|\partial_x\psi| = \partial_x(\partial_x^2\psi/\partial_x\psi)$ when there is no dependence along the y-axis.

For a surface embedded in $\mathbb{R}^3$ depicted by the implicit function $\psi(x,y,z)=0$, the Gaussian curvature $K=\mathscr{R}/2$ satisfies

$$2K = \partial_x\left[\frac{(\partial_x^2\psi+\partial_y^2\psi+\partial_z^2\psi)\partial_x\psi}{(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2}\right] + \partial_y\left[\frac{(\partial_x^2\psi+\partial_y^2\psi+\partial_z^2\psi)\partial_y\psi}{(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2}\right] + \partial_z\left[\frac{(\partial_x^2\psi+\partial_y^2\psi+\partial_z^2\psi)\partial_z\psi}{(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2}\right]$$

$$- (\partial_x^2+\partial_y^2+\partial_z^2)\log\sqrt{(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2}. \tag{19}$$

In particular, for the Monge form $z=f(x,y)$, one has

$$K = \frac{1}{[1+(\partial_x f)^2+(\partial_y f)^2]^2}\det\begin{pmatrix}\partial_x^2 f & \partial_x\partial_y f \\ \partial_y\partial_x f & \partial_y^2 f\end{pmatrix}. \tag{20}$$

Proof Writing $\psi(x,y)$ as a bivariate function of $\zeta=x+iy, \overline{\zeta}=x-iy$, then we have $\partial_\zeta\psi=(\partial_x\psi-i\partial_y\psi)/2$ and $\partial_{\overline{\zeta}}\psi = (\partial_x\psi+i\partial_y\psi)/2$, and both sides of Eq. 18 equal $2\partial_\zeta\partial_{\overline{\zeta}}\log(\partial_\zeta\psi\partial_{\overline{\zeta}}\psi)$.

The right-hand side of Eq. 19 is equivalent to

$$\frac{2}{[(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2]^2(\partial_z\psi)^2}\times$$

$$\times\det\begin{pmatrix}\partial_z\psi(\partial_x^2\psi\partial_z\psi-2\partial_x\psi\partial_x\partial_z\psi)+(\partial_x\psi)^2\partial_z^2\psi & \partial_z\psi(-\partial_x\psi\partial_y\partial_z\psi+\partial_x\partial_y\psi\partial_z\psi-\partial_x\partial_z\psi\partial_y\psi)+\partial_x\psi\partial_y\psi\partial_z^2\psi \\ \partial_z\psi(-\partial_x\psi\partial_y\partial_z\psi+\partial_x\partial_y\psi\partial_z\psi-\partial_x\partial_z\psi\partial_y\psi)+\partial_x\psi\partial_y\psi\partial_z^2\psi & \partial_z\psi(\partial_y^2\psi\partial_z\psi-2\partial_y\psi\partial_y\partial_z\psi)+(\partial_y\psi)^2\partial_z^2\psi\end{pmatrix},$$

which is a standard formula in differential geometry. Taking $\psi(x,y,z)=z-f(x,y)$, we recover Eq. 20. ∎

Remark Despite being a vacuous truth, the differential identity Eq. 18 (when appropriately rewritten) can serve a good purpose for the geometric analysis of two-dimensional problems. See, for example, the applications in §3. □

Example 2.2 (Planar Harmonic Function) *Let $\varphi(x,y)$ be a harmonic function in a certain domain in $\mathbb{R}^2$, then*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|\nabla\varphi(x,y)| = 0.$$

Proof From complex analysis, one can rewrite $|\nabla\varphi(x,y)| = |f'(\zeta)|$ where $\zeta = x + iy$ and $f(\zeta)$ is constructed from $\varphi(x,y)$ and its harmonic conjugate. As $\log|\nabla\varphi(x,y)| = \operatorname{Re}\log f'(\zeta)$ is the real part of a complex analytic function, it must be a two-dimensional harmonic function. ■

Example 2.3 (Conical Harmonic Function) *In spherical coordinates $\mathbf{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\phi)$, the function $\varphi(\mathbf{r}) = h(\tan\frac{\theta}{2}, \phi)$ is harmonic so long as $h(\rho,\phi)$ defines a two-dimensional harmonic function in polar coordinates: $(\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2)h(\rho,\phi) = 0$. The level set of $\varphi(\mathbf{r})$ is a conical surface (necessarily with vanishing scalar curvature) excluding the apex $\mathbf{r} = 0$. The following identity is satisfied for $|\mathbf{r}| \neq 0$:*

$$\Delta\log|\nabla\varphi(\mathbf{r})| = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\log|\nabla\varphi(\mathbf{r})| = 0.$$

Proof With the substitution $\rho = \tan\frac{\theta}{2}$, we obtain

$$(\mathrm{d}\rho)^2 + \rho^2(\mathrm{d}\phi)^2 = \frac{(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\phi)^2}{4\cos^4\frac{\theta}{2}} \quad\text{and}\quad \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} = 4\cos^4\frac{\theta}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

Therefore, we have

$$|\nabla\varphi(\mathbf{r})|^2 = \frac{1}{r^2}\left[\left(\frac{\partial h(\tan\frac{\theta}{2},\phi)}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial h(\tan\frac{\theta}{2},\phi)}{\partial\phi}\right)^2\right] = \frac{1}{4r^2\cos^4\frac{\theta}{2}}\left[\left(\frac{\partial h(\rho,\phi)}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial h(\rho,\phi)}{\partial\phi}\right)^2\right],$$

and

$$\Delta\log|\nabla\varphi(\mathbf{r})| = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\log\frac{1}{2r\cos^2\frac{\theta}{2}}$$

$$+ \frac{1}{4r^2\cos^4\frac{\theta}{2}}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\log\sqrt{\left(\frac{\partial h(\rho,\phi)}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial h(\rho,\phi)}{\partial\phi}\right)^2}$$

$$= \frac{1}{4r^2\cos^4\frac{\theta}{2}}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\log\sqrt{\left(\frac{\partial h(\rho,\phi)}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial h(\rho,\phi)}{\partial\phi}\right)^2}.$$

Now, quoting the result in Example 2.2, we see that $\Delta\log|\nabla\varphi(\mathbf{r})| = 0$. ■

2.2 Examples from Electrostatics

Many worked examples in electrostatics provide analytic expressions for the geometry of equipotential surfaces (level sets of the electrostatic potential $\varphi(\mathbf{r})$, which is a harmonic function) and the corresponding spatial distributions of the electric field intensity $E(\mathbf{r}) = |\nabla\varphi(\mathbf{r})|$. Several specific cases listed below thus serve as direct verifications of the relation $\Delta\log E(\mathbf{r}) + 2K(\mathbf{r}) = 0$, where $K(\mathbf{r}) = \mathscr{R}(\mathbf{r})/2$ is the Gaussian curvature of the equipotential surface.

Example 2.4 (Isolated Ellipsoidal Conductor) *For an isolated conducting ellipsoid surface*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > b > c) \tag{21}$$

with total charge Q, the surface charge is distributed as [Ref. 3, p. 22]

$$\sigma(\mathbf{r}) = \frac{Q}{4\pi abc}\bigg/\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} = \frac{Q}{4\pi}\sqrt[4]{\frac{K(\mathbf{r})}{a^2 b^2 c^2}},$$

thus on the conductor surface (with $\boldsymbol{r}(x,y,z)$ fulfilling Eq. 21), the electrostatic field intensity satisfies $\log E = \log \sqrt[4]{K} +$ const and the following identity is satisfied:

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \log E(\boldsymbol{r}) + \frac{2}{a^2 b^2 c^2} = 0.$$

Proof Following the convention in Ref. [3], we introduce the curvilinear coordinates $\boldsymbol{r} = \boldsymbol{r}(\xi,\eta,\zeta)$ ($\xi \geq -c^2 \geq \eta \geq -b^2 \geq \zeta \geq -a^2$) as the three roots of the following cubic equation in u:

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1.$$

From the argument in [Ref. 3, §4], the electrostatic potential is given by[1]

$$4\pi\epsilon_0 \varphi(\xi,\eta,\zeta) = \frac{Q}{2} \int_\xi^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s+a^2)(s+b^2)(s+c^2)}}$$

and field intensity is distributed as

$$\log E(\xi,\eta,\zeta) = -\log \sqrt{(\xi-\eta)(\xi-\zeta)} + \text{const},$$

and the equipotential surfaces (the $\xi = $ const surfaces), being ellipsoids confocal with the one shown in Eq. 21, has Gaussian curvature

$$K(\xi,\eta,\zeta) = \frac{(a^2+\xi)(b^2+\xi)(c^2+\xi)}{(\xi-\eta)^2(\xi-\zeta)^2}.$$

With the Laplace operator [Ref. 3, p. 19]:

$$\Delta = \frac{4}{(\xi-\eta)(\zeta-\xi)(\eta-\zeta)} \left[(\eta-\zeta)\left(R_\xi \frac{\partial}{\partial\xi}\right)^2 + (\zeta-\xi)\left(R_\eta \frac{\partial}{\partial\eta}\right)^2 + (\xi-\eta)\left(R_\zeta \frac{\partial}{\partial\zeta}\right)^2\right],$$

$$R_u = \sqrt{(u+a^2)(u+b^2)(u+c^2)}, \quad u = \xi,\eta,\zeta$$

one can verify that

$$-2K = -2\frac{(a^2+\xi)(b^2+\xi)(c^2+\xi)}{(\xi-\eta)^2(\xi-\zeta)^2} = \Delta \log E$$

indeed holds. ∎

Remark The ellipsoidal coordinate system employed in the proof becomes ill-defined for the spheroid case (where either $a = b > c$ or $a > b = c$), but the relation $E \propto \sqrt[4]{K}$ and $2K + \Delta \log E = 0$ remains valid, as the readers may check on their own.

In the fully degenerate case $a = b = c$, the equipotential surfaces form a family of concentric spheres. On the surface of radius r, the Gaussian curvature $K = 1/r^2$, and the spatial distribution $\log E(r,\theta,\phi) = -2\log r + \text{const}$. It is trivially true that

$$\Delta \log E = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \log E}{\partial r}\right) = -\frac{2}{r^2} = -2K.$$

For a "point charge" in higher dimensional space $\mathbb{R}^{n+1}$, we have a power law $\log E(\boldsymbol{r}) = -n\log|\boldsymbol{r}| + \text{const}$ instead of the Coulomb inverse square law. Then we may perform the following computations for a hypersphere $\{\boldsymbol{x} \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| = r\}$ of radius r:

$$\Delta \log E = \frac{\partial^2 \log E}{\partial r^2} + \frac{n}{r} \frac{\partial \log E}{\partial r} = -\frac{n(n-1)}{r^2},$$

which is consistent with the fact that a hypersphere of radius r in $\mathbb{R}^{n+1}$ has scalar curvature $\mathscr{R} = n(n-1)/r^2$. □

Below are two more examples concerning conductors at electrostatic equilibrium with external sources.

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[1]We have converted the statements in [Ref. 3, §4] to SI units, wherever applicable.

Example 2.5 (Rectangular Box Conductor Interacting with Electric Charges) *Consider a grounded rectangular box conductor at electrostatic equilibrium with a distribution of electric charges (with density $\varrho(\boldsymbol{r})$) that vanishes in a neighborhood of the box boundary:*

$$\begin{cases} \nabla^2 \varphi(\boldsymbol{r}) = -\frac{\varrho(\boldsymbol{r})}{\epsilon_0}, & \boldsymbol{r} \in \Omega_0 := (0,a) \times (0,b) \times (0,c), \operatorname{supp} \varrho \Subset \Omega_0; \\ \varphi(\boldsymbol{r}) = 0, & \boldsymbol{r} \in \partial\Omega_0. \end{cases}$$

For a boundary point $\boldsymbol{r} \in \partial\Omega_0$ that is not situated on the edges or corners of the rectangular box, we have $\Delta \log |\nabla\varphi(\boldsymbol{r})| = 0$.

Proof The electrostatic potential $\varphi(\boldsymbol{r})$ can be represented as $\epsilon_0 \varphi(\boldsymbol{r}) = \iiint_{\Omega_0} \varrho(\boldsymbol{r}') G_D(\boldsymbol{r},\boldsymbol{r}') \mathrm{d}^3 \boldsymbol{r}'$ where the Dirichlet Green function $G_D(\boldsymbol{r},\boldsymbol{r}')$ is given by

$$G_D(\boldsymbol{r},\boldsymbol{r}') = \frac{8}{abc} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{\ell\pi x}{a} \sin\frac{\ell\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b} \sin\frac{n\pi z}{c} \sin\frac{n\pi z'}{c}}{\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}.$$

Now, at a boundary point $\boldsymbol{r} \in (0,a) \times (0,b) \times \{c\}$ on the top side of the rectangular box, we have $\partial_x^2 \varphi(\boldsymbol{r}) = \partial_y^2 \varphi(\boldsymbol{r}) = \partial_z^2 \varphi(\boldsymbol{r}) = \partial_x \partial_y \varphi(\boldsymbol{r}) = 0$ judging from the properties of $G_D(\boldsymbol{r},\boldsymbol{r}')$. In order to verify that $\Delta \log(E^2) = 0$, it is sufficient to check $E^2 \Delta(E^2) - |\nabla(E^2)|^2 = 0$. As we have

$$E^2 \Delta(E^2) = 2E^2 \operatorname{Tr}\{[\nabla\nabla\varphi(\boldsymbol{r})]^2\} = 4[\partial_z\varphi(\boldsymbol{r})]^2 \left\{[\partial_x\partial_z\varphi(\boldsymbol{r})]^2 + [\partial_y\partial_z\varphi(\boldsymbol{r})]^2\right\}$$

according to the Bochner-Weitzenböck formula in Euclidean space, and

$$|\nabla(E^2)|^2 = \{\partial_x[|\partial_z\varphi(\boldsymbol{r})|^2]\}^2 + \{\partial_y[|\partial_z\varphi(\boldsymbol{r})|^2]\}^2 = 4[\partial_z\varphi(\boldsymbol{r})]^2 \left\{[\partial_x\partial_z\varphi(\boldsymbol{r})]^2 + [\partial_y\partial_z\varphi(\boldsymbol{r})]^2\right\}$$

for $\boldsymbol{r} \in (0,a) \times (0,b) \times \{c\}$, the identity $\Delta \log(E^2) = 0$ is verified. Similarly, we can double-check $\Delta \log |\nabla\varphi(\boldsymbol{r})| = 0$ on the other five faces of the rectangular box. $\blacksquare$

Example 2.6 (Ellipsoid Conductor in Uniform External Field) *Consider a grounded conducting ellipsoid surface with axes $a > b > c$ (defined by Eq. 21), placed in uniform external field $E_0 \boldsymbol{e}_x$. The electrostatic potential [Ref. 3, p. 23]*

$$\varphi(\boldsymbol{r}) = -E_0 \sqrt{\frac{(\xi+a^2)(\eta+b^2)(\zeta+c^2)}{(b^2-a^2)(c^2-a^2)}}[1-F(\xi)], \quad F(\xi) = \frac{\displaystyle\int_\xi^{+\infty} \frac{\mathrm{d}s}{(s+a^2)\sqrt{(s+a^2)(s+b^2)(s+c^2)}}}{\displaystyle\int_0^{+\infty} \frac{\mathrm{d}s}{(s+a^2)\sqrt{(s+a^2)(s+b^2)(s+c^2)}}}$$

satisfies

$$\Delta \log |\nabla\varphi(\boldsymbol{r})| + 2\frac{a^2 b^2 c^2}{\eta^2 \zeta^2} = 0$$

on the conductor surface where $\xi = 0$.

Proof Upon the substitution $F(0) = 1$, we have the following expression for $\xi = 0$:

$$\Delta \log E(\boldsymbol{r}) = \frac{2}{\eta\zeta}\left\{-\frac{a^2 b^2 c^2}{\eta\zeta} - 2a^2 b^2 c^2 \left[\frac{F''(0)}{F'(0)}\right]^2 + 2a^2 b^2 c^2 \frac{F'''(0)}{F'(0)} + a^2(b^2+c^2)\frac{F''(0)}{F'(0)} - \frac{(a^2-3c^2)(b^2+c^2)}{2c^2} + a^2 - 3\frac{b^2 c^2}{a^2}\right\}.$$

With the relations

$$\frac{F''(0)}{F'(0)} = -\frac{3b^2 c^2 + a^2(b^2+c^2)}{2a^2 b^2 c^2}, \quad \frac{F'''(0)}{F'(0)} = \frac{15b^4 c^4 + 6a^2 b^2 c^2(b^2+c^2) + a^4(3b^4 + 2b^2 c^2 + 3c^4)}{4a^4 b^4 c^4},$$

we obtain

$$\Delta \log E(\boldsymbol{r}) = -2\frac{a^2 b^2 c^2}{\eta^2 \zeta^2},$$

which is the claimed identity. $\blacksquare$

3 A Simple Application to p-Harmonic Functions in Low Dimensions

We consider a class of nonlinear partial differential equations

$$\nabla \cdot (|\nabla \psi|^{p-2} \nabla \psi) = 0, \quad p > 1. \tag{22}$$

A solution $\psi \in W^{1,p}(\Omega)$ [4] to the above nonlinear equation minimizes energy functional $\int_\Omega |\nabla \psi|^p \, \mathrm{d}^d \boldsymbol{r}, d > 1$ (which is the L^p analog of the Dirichlet form $\int_\Omega |\nabla \psi|^2 \, \mathrm{d}^d \boldsymbol{r}$) with homogeneous Dirichlet boundary conditions on each connected component of $\partial \Omega$. Functions satisfying Eq. 22 are referred to as p-harmonic functions [5]. By convention, the attention is restricted to the scenario with nowhere vanishing gradients $\nabla \psi(\boldsymbol{r}) \neq \boldsymbol{0}, \boldsymbol{r} \in \Omega$, so that one may recast Eq. 22 into the form of "inhomogeneous Laplace equation"

$$0 = \Delta \psi + (p-2)\nabla \psi \cdot \nabla \log F = \Delta \psi + (p-2)F\frac{\partial F}{\partial \psi}, \quad \text{where } F := |\nabla \psi|. \tag{23}$$

Now that $V = (2-p)F \partial F / \partial \psi$, we immediately arrive at the following geometric identities, according to Eqs. 6 and 11

$$\mathrm{Tr}(\hat{W}) = (1-p)\frac{\partial F}{\partial \psi}, \tag{24}$$

$$\mathcal{R} = -\Delta \log F + (2-p)F^2 \frac{\partial^2 \log F}{\partial \psi^2} + (2-p)^2 \left(\frac{\partial F}{\partial \psi}\right)^2 = F\Delta_\Sigma \frac{1}{F} + (1-p)F^2 \frac{\partial^2 \log F}{\partial \psi^2} + (1-p)(2-p)\left(\frac{\partial F}{\partial \psi}\right)^2$$

$$= F\Delta_\Sigma \frac{1}{F} + (1-p)F\frac{\partial^2 F}{\partial \psi^2} + (1-p)^2 \left(\frac{\partial F}{\partial \psi}\right)^2. \tag{25}$$

Here, Eq. 24 is consistent with Eq. 4.3 in Ref. [5]. From the geometric inequality $(d-2)[\mathrm{Tr}(\hat{W})]^2 - (d-1)\mathcal{R} = (d-1)^2 \mathrm{var}[\mathrm{spec}(\hat{W})] \geq 0$, we obtain

$$\frac{(d-p)(p-1)}{d-1}\left(\frac{\partial F}{\partial \psi}\right)^2 \geq F\Delta_\Sigma \frac{1}{F} + (1-p)F^2 \frac{\partial^2 \log F}{\partial \psi^2}. \tag{26}$$

For $p \neq 2$, we may rewrite Eq. 25 as

$$\mathcal{R} = F\Delta_\Sigma \frac{1}{F} + \frac{1-p}{2-p}F^p \frac{\partial^2(F^{2-p})}{\partial \psi^2} = F\Delta_\Sigma \frac{1}{F} + F^p \frac{\partial}{\partial \psi}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}, \tag{27}$$

and the rightmost expression in the equation above remains valid for $p = 2$.

Now, we may test the validity and usefulness of the formulae above in the following proposition by recovering the $d = 2$ case of Theorem 1.2 in Ref. [5].

Proposition 3.1 (Extremal Properties of Planar p-Harmonic Functions) *Let $\psi(\boldsymbol{r}), \boldsymbol{r} \in \Omega \subset \mathbb{R}^d$ be a p-harmonic function, with $p > 1$. Then its level sets satisfy*

$$\frac{\partial}{\partial \psi}\frac{\mathcal{R}}{F^p} = \frac{1}{F^{p-1}\sqrt{g}}\partial_i \left[\frac{1}{p-1}\left(g^{ij}F^{p-1}\sqrt{g}\partial_j \frac{\mathrm{Tr}(\hat{W})}{F^{p+1}}\right) - \frac{2}{p}\left(g^{ik}b_k^j F^{p-2}\sqrt{g}\partial_j \frac{1}{F^p}\right)\right] + \frac{\partial^2}{\partial \psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}, \tag{28}$$

where (b_k^j) is the matrix representation of the Weingarten transformation $\hat{W}$.

In particular, for $d = 2$, we have the equality for $-F^{1-p}\mathrm{Tr}(\hat{W}) = |\nabla \psi|^{1-p}\kappa$

$$\left(\frac{\Delta_\Sigma}{p-1} + F^2 \frac{\partial^2}{\partial \psi^2}\right)\frac{\kappa}{F^{p-1}} + 2\frac{\kappa}{F^{p-1}}\frac{(p-2)^2}{(p-1)}\left(\frac{\partial \log F}{\partial s}\right)^2 = 0, \tag{29}$$

wherever $\nabla(F^{1-p}\kappa) = \boldsymbol{0}$, so that a non-negative-valued $|\nabla \psi|^{1-p}\kappa$ attains its minimum at the boundary of a domain. (Here in Eq. 29, $\partial/\partial s$ is the tangential derivative with respect to arc length parameter of the level set, and $\Delta_\Sigma = \partial^2/\partial s^2$ is the Laplace-Beltrami operator on the level set.)

Furthermore, for $d = 2$, we have the equality

$$F^\alpha \left(\frac{\Delta_\Sigma}{p-1} + F^2 \frac{\partial^2}{\partial \psi^2}\right)\frac{\kappa}{F^{p-1+\alpha}} = -\frac{\kappa^3}{F^{p-1}}\frac{\alpha(\alpha+2-p)}{(p-1)^2} - \frac{\kappa}{F^{p-1}}\frac{2(p-2)^2 + \alpha(\alpha+3p-6)}{(p-1)}\left(\frac{\partial \log F}{\partial s}\right)^2, \tag{30}$$

wherever $\nabla(F^{1-p-\alpha}\kappa) = \mathbf{0}$. *Thus, if either of the following*

$$\begin{cases} \alpha \in (-\infty, p-2] \cup [0, 2-p] \cup [2(2-p), +\infty), & 1 < p \le 2 \\ \alpha \in (-\infty, 2(2-p)] \cup [2-p, 0] \cup [p-2, +\infty), & p > 2 \end{cases} \tag{31}$$

happens, then a non-negative-valued $|\nabla\psi|^{1-p-\alpha}\kappa$ *attains its minimum at the boundary of a domain. For* $3/2 \le p \le 3$, *the minimum of a non-negative function* $\kappa(\mathbf{r}), \mathbf{r} \in D \cup \partial D$ *is attained at the boundary* ∂D.

Proof We divide both sides of Eq. 27 by F^p and differentiate, to arrive at

$$\frac{\partial}{\partial\psi}\frac{\mathscr{R}}{F^p} = \frac{\partial}{\partial\psi}\left(\frac{1}{F^{p-1}}\Delta_\Sigma\frac{1}{F}\right) + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} = \frac{1}{F^{p-1}\sqrt{g}}\partial_i\frac{\partial}{\partial\psi}\left(g^{ij}\sqrt{g}\partial_j\frac{1}{F}\right) + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}},$$

where the *p*-harmonicity has been exploited in the form of $\partial_\psi(F^{p-1}\sqrt{g}) = 0$. Now, we simplify

$$\frac{\partial}{\partial\psi}\left(g^{ij}\sqrt{g}\partial_j\frac{1}{F}\right) = \frac{1}{p}\frac{\partial}{\partial\psi}\left(g^{ij}F^{p-1}\sqrt{g}\partial_j\frac{1}{F^p}\right)$$

with the relations $\partial_\psi g^{ij} = -2g^{ik}b_k^j/F$ and $\partial_\psi(F^{-p}/p) = F^{-p-1}\mathrm{Tr}(\hat{W})/(p-1)$, to obtain

$$\frac{\partial}{\partial\psi}\left(g^{ij}\sqrt{g}\partial_j\frac{1}{F}\right) = -\frac{2}{p}\left(g^{ik}b_k^j F^{p-2}\sqrt{g}\partial_j\frac{1}{F^p}\right) + \frac{1}{p-1}\left(g^{ij}F^{p-1}\sqrt{g}\partial_j\frac{\mathrm{Tr}(\hat{W})}{F^{p+1}}\right),$$

which proves Eq. 28, a formula not found in Ref. [5].

For $d = 2$, we have $\mathscr{R} \equiv 0$, and $g^{ik}b_k^j = g^{ij}\mathrm{Tr}(\hat{W}) = -g^{ij}\kappa$. This brings us from Eq. 28 to

$$\begin{aligned}
0 &= \frac{1}{F^{p-1}\sqrt{g}}\partial_i\left[\frac{1}{p-1}\left(g^{ij}F^{p-1}\sqrt{g}\partial_j\frac{\mathrm{Tr}(\hat{W})}{F^{p+1}}\right) - \frac{2}{p}\left(g^{ij}\mathrm{Tr}(\hat{W})F^{p-2}\sqrt{g}\partial_j\frac{1}{F^p}\right)\right] + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} \\
&= \frac{1}{F^{p-1}\sqrt{g}}\partial_i\left[\frac{1}{p-1}\left(g^{ij}F^{p-3}\sqrt{g}\partial_j\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\right) - 2\frac{p-2}{p-1}\left(g^{ij}\frac{\mathrm{Tr}(\hat{W})}{F}\sqrt{g}\partial_j\frac{1}{F}\right)\right] + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} \\
&= \frac{1}{F^2}\frac{\Delta_\Sigma(F^{1-p}\mathrm{Tr}(\hat{W}))}{p-1} + \frac{p-3}{(p-1)F^2}\frac{\partial\log F}{\partial s}\frac{\partial}{\partial s}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} + 2\frac{p-2}{p-1}\frac{\mathrm{Tr}(\hat{W})}{F}\frac{\partial}{\partial\psi}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} \\
&\quad - 2\frac{p-2}{(p-1)F^{p-1}}\frac{\partial}{\partial s}\left[\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}F^{p-2}\right]\frac{\partial}{\partial s}\frac{1}{F} + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} \\
&= \left[\frac{1}{F^2}\frac{\Delta_\Sigma(F^{1-p}\mathrm{Tr}(\hat{W}))}{p-1} + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\right] + \left[\frac{3p-7}{(p-1)F^2}\frac{\partial\log F}{\partial s}\frac{\partial}{\partial s}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}} + 2\frac{p-2}{p-1}\frac{\mathrm{Tr}(\hat{W})}{F}\frac{\partial}{\partial\psi}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\right] \\
&\quad + 2\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\frac{(p-2)^2}{(p-1)F^2}\left(\frac{\partial\log F}{\partial s}\right)^2,
\end{aligned} \tag{32}$$

where s is the arc length of the level set, and we have evaluated $\Delta_\Sigma(1/F)$ from Eq. 27 (with $\mathscr{R} = 0$) in the penultimate step.

Now, for any real number α, at points where $\nabla[F^{1-p-\alpha}\mathrm{Tr}(\hat{W})] = \mathbf{0}$, we may transform Eq. 32 into

$$\begin{aligned}
0 &= \left[\frac{1}{F^2}\frac{\Delta_\Sigma(F^{1-p-\alpha}\mathrm{Tr}(\hat{W})F^\alpha)}{p-1} + \frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})F^\alpha}{F^{p-1+\alpha}}\right] + \left[\frac{3p-7}{(p-1)F^2}\frac{\partial\log F}{\partial s}\frac{\partial}{\partial s}\frac{\mathrm{Tr}(\hat{W})F^\alpha}{F^{p-1+\alpha}} + 2\frac{p-2}{p-1}\frac{\mathrm{Tr}(\hat{W})}{F}\frac{\partial}{\partial\psi}\frac{\mathrm{Tr}(\hat{W})F^\alpha}{F^{p-1+\alpha}}\right] \\
&\quad + 2\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\frac{(p-2)^2}{(p-1)F^2}\left(\frac{\partial\log F}{\partial s}\right)^2 \\
&= \left[\frac{F^\alpha\Delta_\Sigma(F^{1-p-\alpha}\mathrm{Tr}(\hat{W})) - \frac{\alpha F^{\alpha+1}\mathrm{Tr}(\hat{W})}{F^{p-1+\alpha}}\Delta_\Sigma\frac{1}{F} + \alpha(\alpha+1)\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\left(\frac{\partial\log F}{\partial s}\right)^2}{F^2(p-1)} + F^\alpha\frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1+\alpha}} + \frac{\mathrm{Tr}(\hat{W})}{F^{p-1+\alpha}}\frac{\partial^2(F^\alpha)}{\partial\psi^2}\right] \\
&\quad + \left[\frac{\alpha(3p-7)}{(p-1)F^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p+1}}\left(\frac{\partial\log F}{\partial s}\right)^2 + 2\frac{p-2}{p-1}\frac{[\mathrm{Tr}(\hat{W})]^2}{F^{p+\alpha}}\frac{\partial(F^\alpha)}{\partial\psi}\right] + 2\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\frac{(p-2)^2}{(p-1)F^2}\left(\frac{\partial\log F}{\partial s}\right)^2 \\
&= \left[\frac{\Delta_\Sigma(F^{1-p-\alpha}\mathrm{Tr}(\hat{W}))}{F^{2-\alpha}(p-1)} + F^\alpha\frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1+\alpha}}\right] + \frac{1}{F^2(p-1)}\left[\frac{\alpha[\mathrm{Tr}(\hat{W})]^2}{F^{p+\alpha-2}}\frac{\partial(F^\alpha)}{\partial\psi} - \frac{\alpha[\mathrm{Tr}(\hat{W})]^2}{F^{2(p-2+\alpha)}}\frac{\partial(F^{2(\alpha-1)+p})}{\partial\psi}\right]
\end{aligned}$$

$$+\left[2\frac{p-2}{p-1}\frac{[\mathrm{Tr}(\hat{W})]^2}{F^{p+\alpha}}\frac{\partial(F^\alpha)}{\partial\psi}\right]+\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\frac{2(p-2)^2+\alpha(\alpha+1)+\alpha(3p-7)}{(p-1)F^2}\left(\frac{\partial\log F}{\partial s}\right)^2$$

$$=\left[\frac{\Delta_\Sigma(F^{1-p-\alpha}\mathrm{Tr}(\hat{W}))}{F^{2-\alpha}(p-1)}+F^\alpha\frac{\partial^2}{\partial\psi^2}\frac{\mathrm{Tr}(\hat{W})}{F^{p-1+\alpha}}\right]-\frac{[\mathrm{Tr}(\hat{W})]^3}{(p-1)^2}\frac{\alpha^2-2\alpha(\alpha-1)-\alpha p+2\alpha(p-2)}{F^{p+1}}$$

$$+\frac{\mathrm{Tr}(\hat{W})}{F^{p-1}}\frac{2(p-2)^2+\alpha(\alpha+1)+\alpha(3p-7)}{(p-1)F^2}\left(\frac{\partial\log F}{\partial s}\right)^2,$$

as claimed in Eq. 30. Solving the inequalities $\alpha(\alpha+2-p)\geq0$ and $2(p-2)^2+\alpha(\alpha+3p-6)\geq0$, we obtain Eq. 31. Setting $\alpha=1-p$ in Eq. 31, we reach the range $3/2\leq p\leq3$. The conclusions drawn from Eqs. 29-31 agree with Ref. [5]. ∎

4 Application to Einstein-Hilbert Actions of High-Dimensional Equipotential Surfaces

During the study of a physical problem that will be described elsewhere, we encountered the following quantity

$$\oint_{\Sigma_\varphi}\mathscr{R}\,\mathrm{d}m(\Sigma_\varphi)$$

which is the Einstein-Hilbert action of a curved equipotential surface Σ_φ (*i.e.* level set of a harmonic function with level value φ). For $\dim(\Sigma_\varphi)=1$, the Einstein-Hilbert action vanishes identically. For $\dim(\Sigma_\varphi)=2$, the Einstein-Hilbert action is equal to 4π times the Euler-Poincaré characteristic $\chi(\Sigma_\varphi)$, so it remains a local constant of φ so long as the field lines are not broken.

In the following proposition, we will present a convexity result for the less trivial cases where $\dim(\Sigma_\varphi)\geq3$. To emphasize that we are handling harmonic functions, we will replace the earlier notation $F=|\nabla\psi|$ with $E=|\nabla\varphi|$. Here, $\varphi(\boldsymbol{r})$ can be regarded as a high-dimensional "electrostatic potential", and $E(\boldsymbol{r})=|\nabla\varphi(\boldsymbol{r})|$ the corresponding "electric field intensity". Still, it will be tacitly assumed that $E=|\nabla\varphi|$ is non-vanishing.

Proposition 4.1 (Conditional Convexity of Einstein-Hilbert Action) *The Einstein-Hilbert action*

$$\mathscr{A}(\varphi):=\oint_{\Sigma_\varphi}\mathscr{R}\,\mathrm{d}m(\Sigma_\varphi)=\int\cdots\int_{\Sigma_\varphi}\mathscr{R}\sqrt{g}\,\mathrm{d}u^1\cdots\mathrm{d}u^n$$

satisfies

$$\frac{\mathscr{A}''(\varphi)}{6}=\oint_{\Sigma_\varphi}\frac{\frac{1}{3}b_m^m G^{ij}b_{ij}-R^{ij}R_{ij}+\frac{1}{2}\mathscr{R}^2}{E^2}\,\mathrm{d}m(\Sigma_\varphi)-\oint_{\Sigma_\varphi}G_{ik}\partial_i\frac{1}{E}\partial_k\frac{1}{E}\,\mathrm{d}m(\Sigma_\varphi).\tag{33}$$

Accordingly, for $n\geq3$, we have $\mathscr{A}''(\varphi)\geq0$ if Σ_φ has non-negative sectional curvature.

Proof It follows directly from Eq. 14* that

$$\mathscr{A}'(\varphi)=-2\oint_{\Sigma_\varphi}\frac{G^{ij}b_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)=2\oint_{\Sigma_\varphi}\frac{G_{ij}(\beta^{ij}-b_\ell^\ell g^{ij})}{E}\,\mathrm{d}m(\Sigma_\varphi)=2\oint_{\Sigma_\varphi}\frac{\beta^{ij}G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)-2\left(1-\frac{n}{2}\right)\oint_{\Sigma_\varphi}\frac{b_\ell^\ell\mathscr{R}}{E}\,\mathrm{d}m(\Sigma_\varphi),$$

where $G_{ij}=R_{ij}-\frac{1}{2}g_{ij}\mathscr{R}$ is the covariant Einstein tensor. Before computing the second-order derivative $\mathscr{A}''(\varphi)$, we need

$$\frac{1}{\sqrt{g}}\frac{\partial(\beta^{ik}\sqrt{g})}{\partial\varphi}=\frac{b_m^m\beta^{ik}}{E}+\frac{\partial}{\partial\varphi}(b_m^m g^{ik}-g^{ij}g^{k\ell}b_{j\ell})$$

$$=\frac{b_m^m\beta^{ik}}{E}-g^{ik}\left(\Delta_\Sigma\frac{1}{E}+\frac{b_m^\ell b_\ell^m}{E}\right)-\frac{2b_m^m g^{ij}b_j^k}{E}+\frac{2g^{im}b_m^j b_j^k}{E}+\frac{2g^{km}b_m^\ell b_\ell^i}{E}-g^{ij}g^{k\ell}\left(\frac{b_j^m b_{m\ell}}{E}-\left(\frac{1}{E}\right)_{;j\ell}\right)$$

$$=\frac{b_m^m(b_\ell^\ell g^{ik}-g^{ij}b_j^k)}{E}-g^{ik}\left(\Delta_\Sigma\frac{1}{E}+\frac{b_m^\ell b_\ell^m}{E}\right)+\left(-\frac{2b_m^m g^{ij}b_j^k}{E}+\frac{2g^{im}b_m^j b_j^k}{E}\right)+\left(\frac{2g^{km}b_m^\ell b_\ell^i}{E}-\frac{g^{ij}b_j^m b_m^k}{E}\right)+\left(\frac{1}{E}\right)^{;ik}$$

$$=-\frac{b_m^m g^{ij}b_j^k}{E}-g^{ik}\left(\Delta_\Sigma\frac{1}{E}-\frac{\mathscr{R}}{E}\right)-\frac{2g^{im}R_m^k}{E}+\frac{g^{km}b_m^\ell b_\ell^i}{E}+\left(\frac{1}{E}\right)^{;ik}$$

$$=-g^{ik}\left(\Delta_\Sigma\frac{1}{E}-\frac{\mathscr{R}}{E}\right)-\frac{3R^{ik}}{E}+\left(\frac{1}{E}\right)^{;ik}=-g^{ik}\left(\Delta_\Sigma\frac{1}{E}+\frac{\mathscr{R}}{2E}\right)-\frac{3G^{ik}}{E}+\left(\frac{1}{E}\right)^{;ik}.\tag{34}$$

Thus, we may evaluate

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{\beta^{ij}G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)=\oint_{\Sigma_\varphi}\frac{G_{ij}}{E}\frac{1}{\sqrt{g}}\frac{\partial(\beta^{ij}\sqrt{g})}{\partial\varphi}\,\mathrm{d}m(\Sigma_\varphi)+\oint_{\Sigma_\varphi}\beta^{ij}\frac{\partial}{\partial\varphi}\frac{G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)$$

$$=\left(\frac{n}{2}-1\right)\oint_{\Sigma_\varphi}\frac{\mathcal{R}}{E}\Delta_\Sigma\frac{1}{E}\,\mathrm{d}m(\Sigma_\varphi)-3\oint_{\Sigma_\varphi}\frac{R^{ij}R_{ij}}{E^2}\,\mathrm{d}m(\Sigma_\varphi)+\frac{5-n}{2}\oint_{\Sigma_\varphi}\frac{\mathcal{R}^2}{E^2}\,\mathrm{d}m(\Sigma_\varphi)$$

$$+\oint_{\Sigma_\varphi}\frac{G_{ik}}{E}\left(\frac{1}{E}\right)^{;ik}\,\mathrm{d}m(\Sigma_\varphi)+\oint_{\Sigma_\varphi}\beta^{ij}\frac{\partial}{\partial\varphi}\frac{G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi).$$

where we have used the squared modulus of the Einstein tensor $G^{ij}G_{ij}=R^{ij}R_{ij}+(\frac{n}{4}-1)\mathcal{R}^2$ in the last step. From Eqs. 13 and 14*, it is not hard to see that

$$\beta^{ij}\frac{\partial}{\partial\varphi}\frac{R_{ij}}{E}=\frac{2G_{ik}}{E}\left(\frac{1}{E}\right)^{;ik}+\frac{\mathcal{R}^2}{E^2}-\frac{b_\ell^\ell}{E\sqrt{g}}\partial_i\left(\beta^{ik}\sqrt{g}\partial_k\frac{1}{E}\right),$$

$$\beta^{ij}\frac{\partial}{\partial\varphi}\frac{\frac{1}{2}g_{ij}\mathcal{R}}{E}=\frac{\mathcal{R}^2}{E^2}+\frac{(n-1)b_m^m}{2}\frac{\partial}{\partial\varphi}\frac{\mathcal{R}}{E}=-\frac{\mathcal{R}^2}{E^2}-\frac{(n-1)b_m^mG^{ij}b_{ij}}{E^2}-\frac{(n-1)b_m^m}{E\sqrt{g}}\partial_i\left(\beta^{ik}\sqrt{g}\partial_k\frac{1}{E}\right),$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{\beta^{ij}G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)=\oint_{\Sigma_\varphi}\frac{G_{ij}}{E}\frac{1}{\sqrt{g}}\frac{\partial(\beta^{ij}\sqrt{g})}{\partial\varphi}\,\mathrm{d}m(\Sigma_\varphi)+\oint_{\Sigma_\varphi}\beta^{ij}\frac{\partial}{\partial\varphi}\frac{G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)$$

$$=\left(\frac{n}{2}-1\right)\oint_{\Sigma_\varphi}\frac{\mathcal{R}}{E}\Delta_\Sigma\frac{1}{E}\,\mathrm{d}m(\Sigma_\varphi)+(n-1)\oint_{\Sigma_\varphi}\frac{b_m^mG^{ij}b_{ij}}{E^2}\,\mathrm{d}m(\Sigma_\varphi)-3\oint_{\Sigma_\varphi}\frac{R^{ij}R_{ij}}{E^2}\,\mathrm{d}m(\Sigma_\varphi)+\frac{5-n}{2}\oint_{\Sigma_\varphi}\frac{\mathcal{R}^2}{E^2}\,\mathrm{d}m(\Sigma_\varphi)$$

$$+\oint_{\Sigma_\varphi}\frac{3G_{ik}}{E}\left(\frac{1}{E}\right)^{;ik}\,\mathrm{d}m(\Sigma_\varphi)+(n-2)\oint_{\Sigma_\varphi}\frac{b_\ell^\ell}{E\sqrt{g}}\partial_i\left(\beta^{ik}\sqrt{g}\partial_k\frac{1}{E}\right)\,\mathrm{d}m(\Sigma_\varphi)$$

as a consequence. Meanwhile,

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{b_\ell^\ell\mathcal{R}}{E}\,\mathrm{d}m(\Sigma_\varphi)=\oint_{\Sigma_\varphi}\frac{b_\ell^\ell}{E}\left[-\frac{2G^{ij}b_{ij}}{E}-\frac{2}{\sqrt{g}}\partial_i\left(\beta^{ik}\sqrt{g}\partial_k\frac{1}{E}\right)\right]\mathrm{d}m(\Sigma_\varphi)+\oint_{\Sigma_\varphi}\mathcal{R}\left(-\Delta_\Sigma\frac{1}{E}+\frac{\mathcal{R}}{E}\right)\mathrm{d}m(\Sigma_\varphi)$$

can be combined into the previous results to yield

$$\frac{\mathscr{A}''(\varphi)}{2}=\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{\beta^{ij}G_{ij}}{E}\,\mathrm{d}m(\Sigma_\varphi)-\left(1-\frac{n}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{b_\ell^\ell\mathcal{R}}{E}\,\mathrm{d}m(\Sigma_\varphi)$$

$$=\oint_{\Sigma_\varphi}\frac{b_m^mG^{ij}b_{ij}}{E^2}\,\mathrm{d}m(\Sigma_\varphi)-3\oint_{\Sigma_\varphi}\frac{R^{ij}R_{ij}}{E^2}\,\mathrm{d}m(\Sigma_\varphi)+\frac{3}{2}\oint_{\Sigma_\varphi}\frac{\mathcal{R}^2}{E^2}\,\mathrm{d}m(\Sigma_\varphi)+\oint_{\Sigma_\varphi}\frac{3G_{ik}}{E}\left(\frac{1}{E}\right)^{;ik}\,\mathrm{d}m(\Sigma_\varphi).$$

After we apply the contracted Bianchi identity $(G_{ik})^{;i}=0$, Eq. 33 emerges.

In the principal curvature coordinates, the Einstein tensor behaves as

$$(G_{ij})_{1\le i,j\le n}=\begin{pmatrix}-g_{11}\sum\limits_{\substack{\ell<m\\\ell\neq1,m\neq1}}k_\ell k_m & & \Large 0\\[2mm] & \ddots & \\[2mm] \Large 0 & & -g_{nn}\sum\limits_{\substack{\ell<m\\\ell\neq n,m\neq n}}k_\ell k_m\end{pmatrix},$$

which is clearly a negative semidefinite matrix if Σ_φ has non-negative sectional curvature. In this case, we then have

$$\frac{\mathscr{A}''(\varphi)}{6}\ge\oint_{\Sigma_\varphi}\frac{\frac{1}{3}b_m^mG^{ij}b_{ij}-R^{ij}R_{ij}+\frac{1}{2}\mathcal{R}^2}{E^2}\,\mathrm{d}m(\Sigma_\varphi). \tag{35}$$

However, we may use principal curvatures to rewrite

$$\frac{1}{3}b_m^mG^{ij}b_{ij}-R^{ij}R_{ij}+\frac{1}{2}\mathcal{R}^2=\sum_{\substack{i<j\\i\neq\ell,j\neq\ell}}k_ik_jk_\ell^2+8\sum_{i<j<\ell<m}k_ik_jk_\ell k_m, \tag{36}$$

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where the second summand is left out for $n = 3$. Such an identity as Eq. 36 is elementary though probably not self-evident. By inspection, the left-hand side of Eq. 36 is a homogeneous quartic polynomial in the variables $(k_1, \ldots, k_n)$, where each variable cannot appear in a power higher than two. Terms like $k_\ell^2 k_m^2$ (where $\ell < m$) will not count towards the net sum, because $-R^{ij} R_{ij}$ contributes $-2k_\ell^2 k_m^2$ while $\frac{1}{2}\mathscr{R}^2$ yields $2k_\ell^2 k_m^2$ in their respective developments. Therefore, by symmetry, the left-hand side of Eq. 36 can be put into the form

$$a \sum_{\substack{i<j \\ i \neq \ell, j \neq \ell}} k_i k_j k_\ell^2 + b \sum_{i<j<\ell<m} k_i k_j k_\ell k_m,$$

for some constants a and b. To determine that $a = 1$, simply set $k_1 = k_2 = k_3 = 1$ and $k_4 = \cdots = k_n = 0$. To find out that $b = 8$, one may consider the special case where $k_1 = \cdots = k_n = 1$.

To summarize, Eqs. 35 and 36 explicitly demonstrate the non-negativity of $\mathscr{A}''(\varphi)$ for equipotential surfaces Σ_φ with non-negative sectional curvature. ∎

Remark In the proof of the proposition above, we have used the following alternative forms of Eqs. 6 and 10:

$$\mathrm{Tr}(\hat{W}) + \frac{\partial E}{\partial \varphi} = 0, \quad \frac{\partial \mathrm{Tr}(\hat{W})}{\partial \varphi} = -\Delta_\Sigma \frac{1}{E} - \frac{[\mathrm{Tr}(\hat{W})]^2 - \mathscr{R}}{E}.$$

Here, the first formula is sensitive to the harmonic condition $\Delta\varphi(\boldsymbol{r}) = 0$, so the conclusion of this proposition does not apply to level sets of arbitrary functions. The second formula displayed above is morally equivalent to the scalar curvature identity $\mathscr{R} + \Delta \log E = 0$, as can be seen from the one-line proof:

$$\mathscr{R} + \Delta \log E = \mathscr{R} - E\Delta_\Sigma \frac{1}{E} + E^2 \frac{\partial^2 \log E}{\partial \varphi^2} = \mathscr{R} - E\Delta_\Sigma \frac{1}{E} - E^2 \frac{\partial}{\partial \varphi} \frac{\mathrm{Tr}(\hat{W})}{E} = \mathscr{R} - E\Delta_\Sigma \frac{1}{E} - E \frac{\partial \mathrm{Tr}(\hat{W})}{\partial \varphi} - E^2 \mathrm{Tr}(\hat{W}) \frac{\partial}{\partial \varphi} \frac{1}{E},$$

which hinges on a degenerate version of Eq. 8. □

References

[1] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov. *Modern Geometry — Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields*, volume 93 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2nd edition, 1992. (Translated by R. G. Burns from the Russian original).

[2] John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, New York, NY, 3rd edition, 1999.

[3] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii. *Electrodynamics of Continuous Media*, volume 8 of *Course of Theoretical Physics*. Butterworth-Heinemann, Oxford, UK, 2nd edition, 1984. (Translated by J. B. Sykes, J. S. Bell and M. J. Kearsley from the Russian original).

[4] Lawrence C. Evans. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic P. D. E. *J. Diff. Eqs.*, 45:356–373, 1982.

[5] Xi-Nan Ma, Qianzhong Ou, and Wei Zhang. Gaussian curvature estimates for the convex level sets of p-harmonic functions. *Comm. Pure Appl. Math.*, 63:935–971, 2010.